Simple Bounds for the Symmetric Capacity of the Rayleigh Fading Multiple Access Channel

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Abstract—**Communication over the i.i.d. Rayleigh slow-fading MAC is considered, where all terminals are equipped with a single antenna. Further, a communication protocol is considered where all users transmit at (just below) the symmetric capacity (per user) of the channel, a rate which is fed back (dictated) to the users by the base station. Tight bounds are established on the distribution of the rate attained by the protocol. In particular, these bounds characterize the probability that the dominant face of the MAC capacity region contains a symmetric rate point, i.e., that the considered protocol strictly attains the sum capacity of the channel. The analysis provides a non-asymptotic counterpart to the diversity-multiplexing tradeoff of the multiple access channel. We then extend this analysis to general multiple-input multiple-output MAC and finally, a practical scheme based on integer-forcing and space-time precoding is shown to be an effective coding architecture for this communication scenario.**

Index Terms—**Multiple access, multiple-input multiple-output (MIMO), Rayleigh channels.**

I. Introduction

In THIS paper we consider communication over the slow (block) fading i.i.d. Rayleigh multiple access channel (MAC). For a given realization of the channel gains, the channel reduces to the classical Gaussian MAC, the capacity region of which is well known, see e.g., [1].

A basic criterion for analyzing the performance of different access methods is the gap from the sum-capacity (the maximal total rate that can be achieved by all the users). We note however, that in many cases, the rate distribution between different users is also of interest and in many applications, fairness is sought and a scheme which provides (maximal) equal rate to all users is desired.

The maximal rate that can be achieved in a system where all users have equal rate is denoted as the symmetric capacity. In case the symmetric and the sum capacity coincide (alternatively the case where the dominant face of the MAC capacity region contains a symmetric-rate point), fairness can be achieved without sacrificing performance. As this is a very desirable working point, it is of interest to investigate what is the probability of this being the case.

Some intuition to that question can be inferred from the diversity multiplexing tradeoff (DMT) of the i.i.d. Rayleigh fading MAC which provides an asymptotic analysis of the symmetric capacity [2]. As we show next, at high values of signal-to-noise ratio (SNR), the symmetric capacity approaches the sum capacity with high probability.

In this paper we characterize the behaviour of the symmetric capacity for finite SNR. From this characterization, the probability of getting fairness for "free" for all SNRs can be easily deduced.

Another motivation for studying the symmetric capacity comes from another design criterion which is the amount of coordination needed by the protocol. High level of coordination results in high throughput loss when finite block length coding is taken into account or increased latency. As the number of users that are simultaneously transmitting increases the amount of coordination increases and thus its impact increases. This is a major issue for new applications being developed for next generation wireless networks (see, e.g., [3], [4]) where supporting high number of users is required along with guaranteeing low latency.

In theory, transmission at rates approaching the symmetric capacity requires minimal coordination; namely a single parameter, the common code rate all users should use. Nonetheless, when it comes to practical schemes that are able to approach this operating point, hitherto practical applicable transmission schemes have relied on a much higher degree of coordination.

Specifically, both time sharing of the points achievable via successive interference cancellation as well as rate splitting are asymmetric between the users and thus require coordination. Furthermore, orthogonal multiple access techniques (e.g., time or frequency division multiple access) also require coordination to achieve its maximal achievable symmetric-rate point which further falls short of the symmetric capacity (unless the latter coincides with the sum capacity).

The contribution of the present work is two-fold:

1) Establishing (statistical) bounds on the gap between the symmetric capacity and sum capacity for the Rayleigh-fading MAC.
2) Proposing a practical scheme that is able to approach the symmetric capacity with the minimal possible degrees of coordination. i.e., specification of the common per-user transmission rate.

These two points have immediate practical implications. Specifically, we are able to characterize the performance of a protocol where all users transmit at a rate just below the

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symmetric capacity (per user) of the channel. The underlying assumption is that the latter rate is dictated to the users by the base station, utilizing a minimal amount of feedback (which does not scale with the number of users).

Our first result is an exact characterization of the performance of the suggested communication protocol, when assuming an optimal (maximum-likelihood) receiver, for the two-user case where all nodes are equipped with a single antenna. We then extend the analysis to the scenario of an N-user Rayleigh-fading MAC where all nodes are equipped with a single antenna. For this scenario, we provide inner and outer bounds on performance. We then further extend the analysis to a general symmetric i.i.d. Rayleigh multiple-input multiple-output (MIMO) MAC.

The derived tight characterization of the distribution of the symmetric capacity can serve as a basis for deriving other figures of merit (such as the ergodic capacity or the outage probability for any target rate). It is worthwhile noting that although the problem studied is by now quite classical, and the derivation relies only on elementary techniques, the obtained results–given in the from of simple closed-form expressions–appear to have eluded previous studies.

Since the complexity of maximum-likelihood (ML) receiver is prohibitive, we also consider the performance attained by a practical integer-forcing (IF) receiver, demonstrating that it performs quite well in the considered scenario. Interestingly, we observe that in order to approach the symmetric capacity with an IF receiver, another lesson from the MIMO-MAC DMT analysis should be followed. Specifically, it is necessary to apply "space-time" precoding at the transmitters (see, e.g., [5]). We note that other coding approaches have been proposed to attain the goal of requiring minimal coordination, see, e.g., [6] and [7].

The rest of this paper is organized as follows. Section II provides the problem formulation. Section III recounts the DMT of the i.i.d. Rayleigh-fading MIMO-MAC. As mentioned above, this asymptotic analysis provides intuition and tools that are subsequently refined to a full characterization of the considered communication protocol. In Section IV, the performance of the protocol is analyzed for the case where all terminals are equipped with a single antenna. In Section V, bounds are derived for the general case of N users, where each user has N_t antennas and the receiver is equipped with N_r antennas. In Section VI, it is demonstrated that an IF receiver combined with (structured or random) space-time precoding yields performance that is close to the established theoretical limits of the proposed communication protocol. Finally, Section VII concludes the paper.

II. Problem Formulation and Preliminaries

To simplify derivations, we will assume throughout that all users are equipped with the same number of transmit antennas. The results can easily be extended to a more general scenario.

Accordingly, we consider a MIMO-MAC with N users, where each transmitter has N_t antennas and the receiver is equipped with N_r antennas. The channel model can be expressed as

$$\mathbf{y} = \sum_{i=1}^{N} \mathbf{H}_i \mathbf{x}_i + \mathbf{n} \tag{1}$$

where $\mathbf{H}_i$ is the channel matrix between user i and the receiver. We assume an i.i.d. Rayleigh-fading model so that $\mathbf{H}_i \sim \mathcal{CN}(0, \mathsf{SNR} \cdot \mathbf{I}_{N_r})$ and $\mathbf{n} \sim \mathcal{CN}(0, \mathbf{I}_{N_r})$, where there is no statistical dependence over space nor time.[1] We assume that the transmitted data $\mathbf{x}_i \in \mathbb{C}^{N_t \times 1}$ is isotropic ("white") for each user and that all users are subject to the same power constraint P where the SNR is absorbed in the channel gains. We assume that channel state information (CSI) is available at the receiver and we analyze a transmission scheme where the sum-capacity (defined later) is communicated to the transmitters.

Define a subset of users by $\mathcal{S} \subseteq \{1, 2, \ldots, N\}$. Then, the capacity region of the channel is given by (see, e.g., [1]) all rate vectors $(R_1, \ldots, R_N)$ satisfying

$$\begin{aligned}\sum_{i \in \mathcal{S}} R_i &\leq C(\mathcal{S}) \\ &\triangleq \log\det\left(\mathbf{I}_{N_r} + \sum_{i \in \mathcal{S}} \mathbf{H}_i \mathbf{H}_i^H\right),\end{aligned} \tag{2}$$

for all subsets $\mathcal{S}$ in the power set of $\{1, 2, \ldots, N\}$. The sum capacity is given by

$$C \triangleq C(\{1, 2, \ldots, N\}) \tag{3}$$

$$= \log\det\left(\mathbf{I}_{N_r} + \sum_{i=1}^{N} \mathbf{H}_i \mathbf{H}_i^H\right). \tag{4}$$

If we impose the constraint that all users transmit at the same rate, then the maximal achievable rate is given by substituting $R_i = C_{\Sigma-\mathrm{sym}}/N$ in (2), from which it follows that the symmetric capacity $C_{\Sigma-\mathrm{sym}}$ is dictated by the bottleneck:

$$C_{\Sigma-\mathrm{sym}} = \min_{\mathcal{S} \subseteq \{1,2,\ldots,N\}} \frac{N}{|\mathcal{S}|} \log\det\left(\mathbf{I}_{N_r} + \sum_{i \in \mathcal{S}} \mathbf{H}_i \mathbf{H}_i^H\right). \tag{5}$$

We study the conditional "cumulative distribution function":[2]

$$\Pr(C_{\Sigma-\mathrm{sym}} < R | C). \tag{6}$$

The latter quantity provides a full statistical characterization of the performance of the transmission protocol considered. Another interpretation of (6) is as a conditional outage probability in an open-loop scenario; that is, in a scenario where all users (when they are active) transmit at a common target rate R. For a given number of active users N, the outage probability is then given by $\mathbb{E}[\Pr(C_{\Sigma-\mathrm{sym}} < N \cdot R | C)]$ where the expectation is over C and is computed w.r.t. an i.i.d. Rayleigh distribution.

[1] The time index remains implicit since it plays no role in the analysis. Of course, coding over large blocklength is needed to approach the information-theoretic limits.

[2] We use quotation marks since we impose strict inequality in $C_{\Sigma-\mathrm{sym}} < R$.

III. Lessons From the DMT

Some insight into the performance of the considered protocol may be obtained by considering the DMT of the symmetric Rayleigh-fading MIMO-MAC channel, which was studied in [2]. As a special case, the scenario where all users transmit at the same rate was considered in detail, for which a simple expression for the DMT was derived.

We first recall the basic definitions of the DMT framework. A scheme $\{C(\mathsf{SNR})\}$ is a family of codes, one at each SNR level (and single coherence block). Let $R(\mathsf{SNR})$ and $P_e(\mathsf{SNR})$ denote their data rate (in bits per symbol period) and the ML probability of detection error, respectively.

Definition 1: Scheme $\{C(\mathsf{SNR})\}$ is said to achieve spatial multiplexing gain r and diversity gain d if the data rate

$$\lim_{\mathsf{SNR}\to\infty} \frac{R(\mathsf{SNR})}{\log \mathsf{SNR}} \geq r,$$

and the average error probability

$$\lim_{\mathsf{SNR}\to\infty} \frac{\log P_e(\mathsf{SNR})}{\log \mathsf{SNR}} \leq -d.$$

For each r, define $d^*_{N_t,N_r}(r)$ to be the supremum of the diversity gain achieved over all schemes. Equivalently, for each d, define $r^*_{N_t,N_r}(d)$ to be the supremum of the multiplexing gain achieved over all schemes.

In [8], it was shown that the DMT of the i.i.d. single-user Rayleigh-fading MIMO channel with N_t transmit antennas and N_r receive antennas (provided that the block length $l \geq N_t + N_r + 1$)) is a piecewise linear curve such that $d^*_{N_t,N_r}(r) = (N_t - r)(N_r - r)$ for every integer $r \leq \min(N_t, N_r)$.

In [2], the DMT of the Rayleigh MIMO-MAC with N users, where each transmitter has N_t antennas and the receiver has N_r antennas, and where all users transmit at the same rate, was shown as

$$d^*_{\mathrm{sym}}(r) = \begin{cases} d^*_{N_t,N_r}(r), & r \leq \min(N_t, \dfrac{N_r}{N+1}) \\ d^*_{N\cdot N_t,N_r}(N\cdot r), & r \geq \min(N_t, \dfrac{N_r}{N+1}) \end{cases} \quad (7)$$

where $d^*_{N_t,N_r}(r)$ is the DMT of the i.i.d. single-user Rayleigh-fading MIMO channel with N_t transmit antennas and N_r receive antennas (provided that the block length $l \geq N_t + N_r + 1$); see, e.g., [8]).

Although the DMT analysis is asymptotic in nature (and assumes no channel state information at the transmitter), instructive lessons may nonetheless be drawn from it. First, it is clear that in the limit of high SNR, the ratio of the symmetric capacity and sum capacity approaches one in probability (since the DMT is strictly positive for any multiplexing gain smaller than the maximal attainable degrees of freedom).

More importantly, the analysis of the typical error events in the Rayleigh-fading MAC (with equal-rate transmission) reveals that with high probability, outage occurs either as if all users were considered as a single one ("antenna pooling") or as a result of a single-user constraint constituting the bottleneck [2]. These two regimes are reflected in the two cases appearing in (7).

Further, it can be easily shown that for a scalar MAC ($N_r = N_t = 1$) with two or more users, the antenna pooling

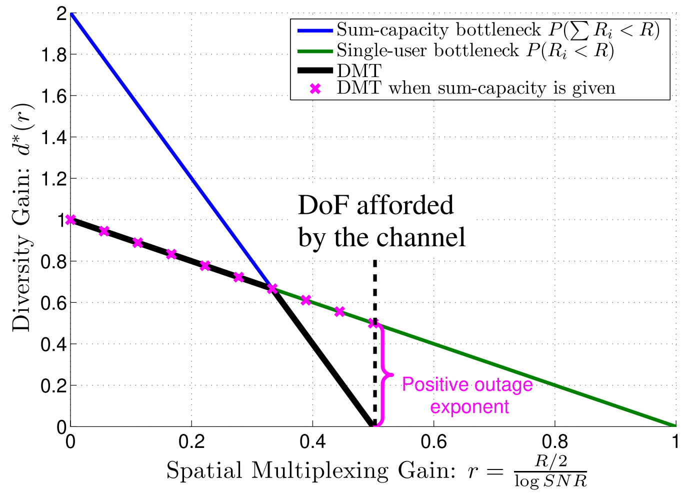

Fig. 1. DMT curve for a two-user scalar Rayleigh-fading MAC where all terminals are equipped with a single antenna.

bottleneck amounts to the probability that the sum-capacity is below the target rate.[3] As for a (symmetric) transmission protocol where the target rate is set to just below the sum capacity, the latter type of outage event cannot occur. It follows that the diversity gain at the maximal multiplexing gain (the maximal attainable degrees of freedom) is *strictly positive*. This in turn implies that the ratio between the symmetric capacity and the sum capacity will approach 1 quite fast as the SNR grows.

The DMT for two users is depicted in Figure 1 where the two bottlenecks mentioned above are shown: the single-user bottleneck and the "antenna pooling" one, the latter which will also be referred to as the sum-capacity bottleneck. We adopt the notation of [8] and plot the diversity as a function of a normalized multiplexing-gain per user. As detailed in [9], the number of degrees of freedom afforded by the channel is the minimum between the "effective" number of transmit and receiver antennas.

In fact, in the case of two users, we show that perfect fairness may be gained "for free" with high probability. This is, the probability that the symmetric capacity is equal to the sum capacity approaches 1 rather fast as a function of the SNR; hence, validating the intuition gained from the DMT.

IV. I.I.D. Rayleigh-Fading MAC With Single-Antenna Terminals

When all terminals are equipped with a single antenna, the Rayleigh-fading MAC is described by

$$y = \sum_{i=1}^{N} h_i x_i + n \quad (8)$$

[3]Note that outage probability is defined (Equation 9 in [9]) to (exponentially) equal the probability that the normalized (isotopic) mutual information is below that target rate. We further note that when considering the antenna pooling regime, this normalization amounts to calculating the sum-capacity of the MIMO-MAC.

and the symmetric capacity is given by

$$C_{\Sigma-\mathrm{sym}} = \min_{\mathcal{S}\subseteq\{1,2,\ldots,N\}} \frac{N}{|\mathcal{S}|}\log\left(1+\sum_{i\in\mathcal{S}}|h_i|^2\right). \tag{9}$$

A. Two-User i.i.d. Single Antenna Rayleigh-Fading MAC

We begin by analyzing the simplest case of a two-user scalar MAC, for which we obtain an exact characterization of (6).

Theorem 1: For a two-user i.i.d. Rayleigh-fading MAC with sum capacity C, for any rate $R \le C$,

$$Pr(C_{\Sigma-\mathrm{sym}} < R|C) = 2\cdot\frac{2^{R/2}-1}{2^C-1}. \tag{10}$$

Proof: The sum capacity of two-user i.i.d. Rayleigh-fading is

$$C = \log(1+|h_1|^2+|h_2|^2) \tag{11}$$

This means that given C, we have $|h_1|^2+|h_2|^2 = 2^C-1$. Equivalently, this suggests that given C, $\mathbf{h} \triangleq (h_1,h_2)$ is uniformly distributed over a two-dimensional complex sphere of radius $\sqrt{2^C-1}$. Hence, $\mathbf{h}/\|\mathbf{h}\|$ can be viewed as the first row of a random (Haar) unitary matrix $\mathbf{U}$.

By (9), we obtain

$$C_{\Sigma-\mathrm{sym}} = \min\{2C(\{1\}), 2C(\{2\}), C\}. \tag{12}$$

We start by analyzing $2C(\{1\})$, which is given by

$$\begin{aligned}2C(\{1\}) &= 2\log\left(1+|h_1|^2\right)\\ &= 2\log\left(1+|\mathbf{U}_{1,1}|^2(2^C-1)\right).\end{aligned} \tag{13}$$

It follows that

$$\begin{aligned}\Pr(2C(\{1\})<R|C) &= \Pr\left(|\mathbf{U}_{1,1}|^2 < \frac{2^{R/2}-1}{2^C-1}\right)\\ &= \Pr\left(|\mathbf{U}_{1,1}|^2 \in \left[0, \frac{2^{R/2}-1}{2^C-1}\right)\right)\end{aligned} \tag{14}$$

Since (see, e.g., [10]) for a 2×2 matrix drawn uniformly with respect to the Haar measure, we have $|\mathbf{U}_{1,1}|^2 \sim \mathrm{Unif}([0,1])$, it follows that

$$\Pr(2C(\{1\})<R|C) = \frac{2^{R/2}-1}{2^C-1}. \tag{15}$$

Now, since $\mathbf{U}_{1,1}$ and $\mathbf{U}_{1,2}$ are the elements of a row in a unitary matrix, we have

$$|\mathbf{U}_{1,1}|^2+|\mathbf{U}_{1,2}|^2 = 1. \tag{16}$$

Hence,

$$\begin{aligned}\Pr(2C(\{2\})<R|C) &= \Pr\left(|\mathbf{U}_{1,2}|^2 < \frac{2^{R/2}-1}{2^C-1}\right)\\ &= \Pr\left(1-|\mathbf{U}_{1,1}|^2 < \frac{2^{R/2}-1}{2^C-1}\right)\\ &= \Pr\left(|\mathbf{U}_{1,1}|^2 \in \left(1-\frac{2^{R/2}-1}{2^C-1}, 1\right]\right)\end{aligned} \tag{17}$$

Since for any rate $R\le C$, the intervals appearing in (14) and (17) are disjoint and of the same length, it follows that

$$\Pr(C_{\Sigma-\mathrm{sym}} < R|C) = 2\cdot\frac{2^{R/2}-1}{2^C-1}. \tag{18}$$

■

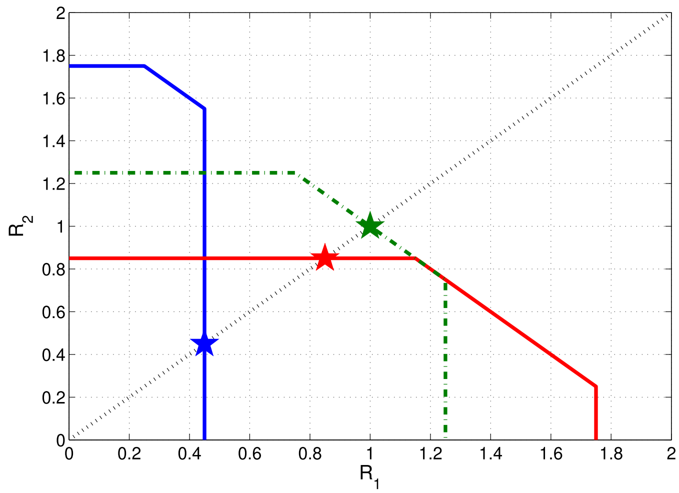

Fig. 2. Different capacity regions corresponding to a two-user MAC with sum capacity $C=2$. For the channel depicted with a dashed-dotted line, the dominant face constitutes the bottleneck and $C_{\Sigma-\mathrm{sym}} = C$.

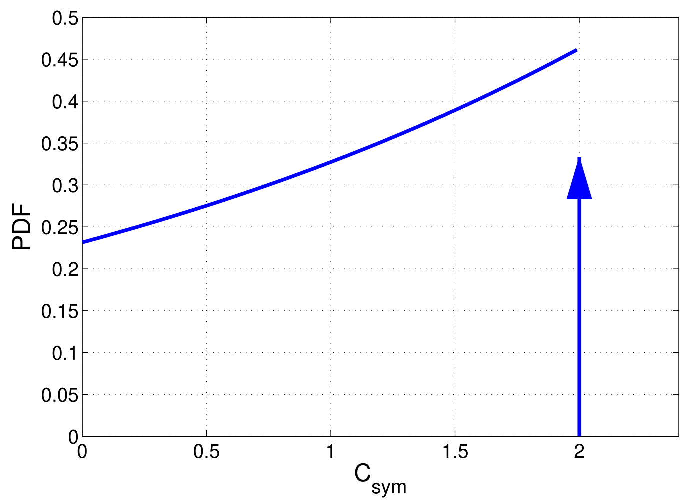

Fig. 3. Probability density function of the symmetric capacity of a two-user i.i.d. Rayleigh-fading MAC given that the sum capacity is $C=2$.

We note that the probability in (18) is strictly smaller than 1 at $R=C$. Thus, the probability that the symmetric capacity coincides with the sum capacity is strictly positive.

Figure 2 depicts the capacity region for three different channel realizations for which the sum capacity equals 2. The probability that the symmetric capacity coincides with the sum capacity amounts to the probability that the symmetric rate line passes through the dominant face of the capacity region and is given by

$$\begin{aligned}\Pr\left(C_{\Sigma-\mathrm{sym}} = C|C\right) &= 1-\Pr\left(C_{\Sigma-\mathrm{sym}} < C|C\right)\\ &= 1-2\cdot\frac{2^{C/2}-1}{2^C-1}.\end{aligned} \tag{19}$$

As an example, for $C=2$, this probability is $1/3$.

Figure 3 depicts the probability density function of the symmetric capacity of a two-user i.i.d. Rayleigh-fading MAC given that the sum capacity is $C=2$. The probability in (19) manifests itself as a delta function at the sum capacity.

B. Extension to the N-User i.i.d. Scalar Rayleigh-Fading MAC

Theorem 1 may be extended to the case of $N > 2$ users. However, rather than obtaining an exact characterization of the distribution of the symmetric capacity, we will now be content with deriving lower and upper bounds for it.

Let us define

$$P_{\text{out}}(k, R|C) \triangleq \Pr\left(\frac{N}{k}C(\mathcal{S}) < R \middle| C\right) \quad (20)$$

We begin with the following lemma which is the key technical step from which Theorem 2 follows.

Lemma 1: For an N-user i.i.d. Rayleigh-fading MAC with sum capacity C, and for any subset of users $\mathcal{S} \subseteq \{1, 2, \ldots, N\}$ with cardinality k, we have

$$P_{\text{out}}(k, R|C) = \frac{\mathcal{B}(\frac{2^{R|\mathcal{S}|/N}-1}{2^C-1}; |\mathcal{S}|, N-|\mathcal{S}|)}{\mathcal{B}(1; |\mathcal{S}|, N-|\mathcal{S}|)}$$

where $0 \le R \le C$,

$$\mathcal{B}(x; a, b) = \int_0^x u^{a-1}(1-u)^{b-1} du$$

is the incomplete beta function (where $0 \le x \le 1$)

Proof: Similar to the case of two users, $\mathbf{h} \triangleq (h_1, \ldots, h_N)$ is uniformly distributed over an N-dimensional complex sphere of radius $\sqrt{2^C - 1}$ and hence $\mathbf{h}/\|\mathbf{h}\|$ may be viewed as the first row of a unitary matrix $\mathbf{U}$ drawn at random according to the Haar measure.

By symmetry, for any set $\mathcal{S}$ with cardinally k, the distribution of $C(\mathcal{S})$ is equal to that of

$$\begin{aligned} C(\{1, 2, \ldots, k\}) &= \log\left(1 + \sum_{i=1}^{k} |h_i|^2\right) \\ &= \log\left(1 + \left(2^C - 1\right)\sum_{i=1}^{k} |U_{1,i}|^2\right). \quad (21) \end{aligned}$$

Denoting the partial sum of k entries as $X = \sum_{i=1}^{k} |\mathbf{U}_{1,i}|^2$, we therefore have

$$\begin{aligned} \Pr\left(\frac{N}{k}C(\mathcal{S}) < R \middle| C\right) &= \Pr\left(1 + \left(2^C - 1\right)X < 2^{R\frac{k}{N}}\right) \\ &= \Pr\left(X < \frac{2^{R\frac{k}{N}} - 1}{2^C - 1}\right). \quad (22) \end{aligned}$$

We note that the vector $(|\mathbf{U}_{1,1}|^2, \ldots, |\mathbf{U}_{1,N}|^2)$ follows the Dirichlet distribution and a partial sum of its entries has a Jacobi (also referred to as MANOVA) distribution. To see this, we note that (21) can be written as

$$\frac{N}{k}C(\{1, 2, \ldots, k\}) = \frac{N}{k}\log\left(1 + (2^C - 1)\mathbf{U}(k)_1\mathbf{U}(k)_1^H\right) \quad (23)$$

where $\mathbf{U}(k)_1$ is a vector which contains the first k elements of the first row of $\mathbf{U}$. Noting that since $\mathbf{U}(k)_1$ is a submatrix of a unitary matrix, its singular values follow (see, e.g., [11]) the Jacobi distribution, and more specifically, X has Jacobi distribution with rank 1. We thus obtain

$$\begin{aligned} \Pr\left(\frac{N}{k}C(\mathcal{S}) < R \middle| C\right) &= \int_0^{\frac{2^{Rk/N}-1}{2^C-1}} x^{k-1}x^{N-k-1} d\lambda \\ &= \frac{\mathcal{B}\left(\frac{2^{Rk/N}-1}{2^C-1}; k, N-k\right)}{\mathcal{B}(1; k, N-k)}, \end{aligned}$$

where $\mathcal{B}(x; a, b)$ is the incomplete beta function defined. ∎

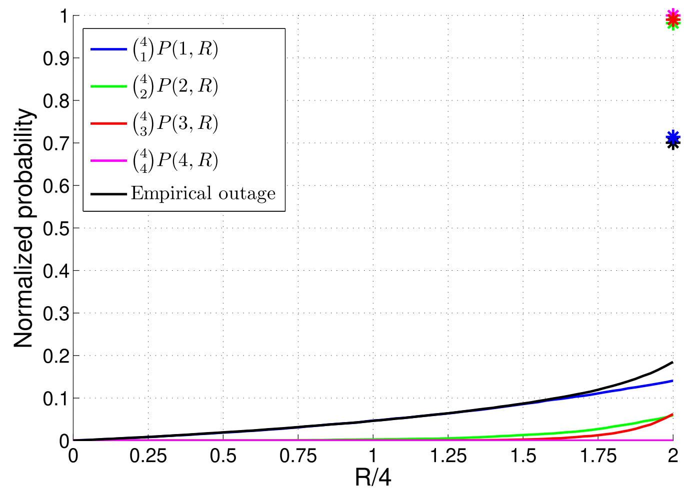

Fig. 4. Demonstration of the quantities appearing in the bounds appearing in Theorem 2 for the case of a 4-user i.i.d. Rayleigh-fading MAC with sum capacity $C = 8$.

Theorem 2: For an N-user scalar i.i.d. Rayleigh-fading MAC, we have

$$\begin{aligned} \max_k P_{\text{out}}(k, R|C) &\le Pr\left(C_{\Sigma-\text{sym}} < R|C\right) \\ &\le \sum_{k=1}^{N} \binom{N}{k} P_{\text{out}}(k, R|C), \quad (24) \end{aligned}$$

where $P_{\text{out}}(k, R|C)$ is defined in (20) *and given in Lemma 1.*

Proof: To establish the left hand side of the theorem, first note that $C_{\Sigma-\text{sym}} \le C(\mathcal{S})$ for any subset $\mathcal{S}$ and hence

$$C_{\Sigma-\text{sym}} \le \min_k \frac{N}{k}C(\{1, 2, \ldots, k\}). \quad (25)$$

It follows that

$$\begin{aligned} &\Pr\left(C_{\Sigma-\text{sym}} < R \middle| C\right) \\ &\ge \Pr\left(\min_k \frac{N}{k}C(\{1, 2, \ldots, k\}) < R \middle| C\right) \\ &= \Pr\left(\bigcup_k \left\{\frac{N}{k}C(\{1, 2, \ldots, k\}) < R\right\} \middle| C\right) \\ &\ge \max_k \Pr\left(\frac{N}{k}C(\{1, 2, \ldots, k\}) < R \middle| C\right) \\ &= \max_k P_{\text{out}}(k, R|C). \quad (26) \end{aligned}$$

The right hand side follows by the union bound. ∎

Figures 4 and 5 illustrate the theorem for the case of four users (where the markers indicate the height of the delta functions). As can be seen from Figure 4, already at not very

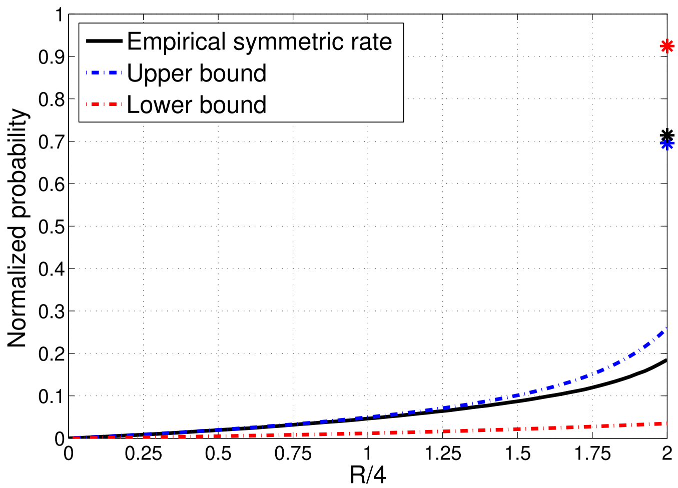

Fig. 5. Comparison of empirical evaluation of (9) and Theorem 2 (upper and lower bounds for the outage probability) for a 4-user i.i.d. Rayleigh-fading MAC ($N_t = N_r = 1$) with sum capacity $C = 8$.

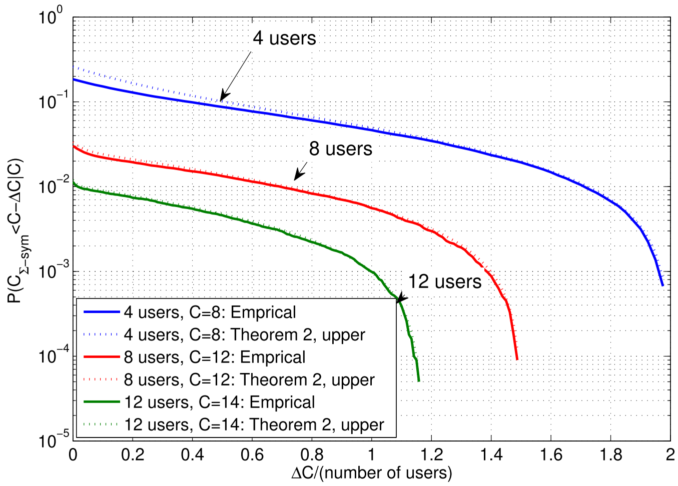

Fig. 6. Comparison of empirical outage and the upper bound of Theorem 2 as the number of users (each equipped with a single antenna) varies. The ratio between the sum-capacity and $\log(\text{number of users})$ is close to 4.

high values of capacity, the single-user constraints already constitute the bottleneck. We further observe from Figure 5 that the union bound is quite tight.

When comparing bounds and performance as the number of users varies, it is natural to scale the sum-capacity logarithmically with the number of users. The reason is that the (expected) sum capacity will grow in this manner assuming each user brings its own power. As mentioned above (when describing intuition gained from DMT analysis), as the sum-capacity increases, the single-user constraint becomes ever more the dominant error event. With regards to the upper bound in Theorem 2, this intuition suggests that the upper bound should become tighter as the number of users grows (as the combinatorial coefficient corresponding to a single-user constraint is only linear in the number of users).

Figure 6 provides a comparison between the empirical outage probability and the upper bound of Theorem 2 when the ratio between the sum-capacity and the number of users is close to 4. As can be seen, as the number of users grows (and each user "brings its own" power) the probability that the symmetric capacity is close to the sum-capacity increases. Further, the gap from the empirical probability and the upper bound decreases.

V. Upper Bound on the Outage Probability for the Symmetric N-User Rayleigh-Fading MIMO MAC

We now consider the symmetric MIMO-MAC scenario where each of the N users is equipped with N_t antennas and the receiver is equipped with N_r antennas. In this case, the channel as described by (1) can be rewritten as

$$\mathbf{y} = \mathcal{H}\mathcal{X} + \mathbf{n} \tag{27}$$

where

$$\mathcal{H} = \begin{bmatrix}\mathbf{H}_1 & \mathbf{H}_2 & \dots & \mathbf{H}_N\end{bmatrix}$$

and

$$\mathcal{X} = \begin{bmatrix}\mathbf{x}_1^T & \mathbf{x}_2^T & \dots & \mathbf{x}_N^T\end{bmatrix}^T.$$

Therefore, the symmetric capacity (5) can be expressed as

$$C_{\Sigma-\text{sym}} = \min_{\mathcal{S}\subseteq\{1,2,\dots,N\}} \frac{N}{|\mathcal{S}|}\log\det\left(\mathbf{I} + \mathcal{H}_{\mathcal{S}}^H\mathcal{H}_{\mathcal{S}}\right) \tag{28}$$

where $\mathcal{H}_{\mathcal{S}}$ is defined as the submatrix of $\mathcal{H}_{\mathcal{S}}$ generated from taking only the channel matrices $\mathbf{H}_i$ corresponding to user indices i such that $i \in \mathcal{S}$.

In order to leverage the bounds derived for the scalar MAC scenario, we may use the simple bound (see, e.g. [12], Equation (5))

$$\log\det\left(\mathbf{I} + \mathcal{H}_{\mathcal{S}}^H\mathcal{H}_{\mathcal{S}}\right) \geq \log\left(1 + \|\mathcal{H}_{\mathcal{S}}\|_F^2\right), \tag{29}$$

where $\|\mathbf{A}\|_F$ denotes the Frobenius norm of a matrix $\mathbf{A}$. Denote the "Frobenius-norm mutual information" by

$$\tilde{C}(\mathcal{S}) = \log\left(1 + \|\mathcal{H}_{\mathcal{S}}\|_F^2\right) \tag{30}$$

and

$$\tilde{C} = \log\left(1 + \|\mathcal{H}\|_F^2\right). \tag{31}$$

It follows that for any channel realization $C \geq \tilde{C}$ and similarly, for any subset of users $\mathcal{S}$, we have $C(\mathcal{S}) \geq \tilde{C}(\mathcal{S})$.

Considering now the performance of a protocol where all users transmit at a rate that is just below $\tilde{C}/N$, the counterparts of Lemma 1 and Theorem 2 are the following.

Lemma 2: For a symmetric N-user $N_r \times N_t$ Rayleigh-fading MIMO-MAC with Frobenius-norm mutual information $\tilde{C}$, for any subset of users $\mathcal{S} \subseteq \{1, 2, \dots, N\}$ with cardinality k, we have

$$\begin{aligned} Pr&\left(\frac{N}{|\mathcal{S}|}\tilde{C}(\mathcal{S}) < R|\tilde{C}\right) \\ &= \frac{\mathcal{B}(\frac{2^{R|\mathcal{S}|/N}-1}{2^{\tilde{C}}-1}; |\mathcal{S}|N_rN_t, (N-|\mathcal{S}|)N_rNt)}{\mathcal{B}(1; |\mathcal{S}|N_rN_t, (N-|\mathcal{S}|)N_rN_t)} \\ &\triangleq \tilde{P}_{\text{out}}(k, R|\tilde{C}) \end{aligned} \tag{32}$$

where $\mathcal{B}(x; a, b)$ is the incomplete beta function, $\tilde{C}(\mathcal{S})$ is defined in (30) *and $\tilde{C}$ is defined in* (31).

Proof: Denoting by $\mathbf{h}_{\text{vec}}$ the vectorization of $\mathcal{H}$, we have

$$\tilde{C} = \log\left(1 + \sum_{i=1}^{N_r N_t N} |h_{\text{vec,i}}|^2\right). \quad (33)$$

As noted in the previous section, conditioned on $\tilde{C}$, $\mathbf{h}_{\text{vec}} \triangleq (h_1, \ldots, h_{N_r N_t N})$ is uniformly distributed over an $N_r N_t N$-dimensional complex sphere of radius $\sqrt{2^{\tilde{C}} - 1}$ and hence $\mathbf{h}_{\text{vec}}/\|\mathbf{h}_{\text{vec}}\|$ may be viewed as the first row of a unitary matrix $\mathbf{U}$ drawn at random according to the Haar measure.

By symmetry, for any set $\mathcal{S}$ with cardinally k, the distribution of $\tilde{C}(\mathcal{S})$ is equal to that of

$$\begin{aligned}\tilde{C}(\{1,2,\ldots,k\}) &= \log\left(1 + \sum_{i=1}^{N_r N_t k} |h_{\text{vec,i}}|^2\right) \\ &= \log\left(1 + \left(2^{\tilde{C}} - 1\right)\sum_{i=1}^{N_r N_t k} |U_{1,i}|^2\right). \quad (34)\end{aligned}$$

Denoting the partial sum of k entries as $X = \sum_{i=1}^{N_r N_t k} |U_{1,i}|^2$, we therefore have

$$\begin{aligned}\Pr\left(\frac{N}{k}\tilde{C}(\mathcal{S}) < R \middle| \tilde{C}\right) &= \Pr\left(1 + \left(2^{\tilde{C}} - 1\right) X < 2^{R\frac{k}{N}}\right) \\ &= \Pr\left(X < \frac{2^{R\frac{k}{N}} - 1}{2^{\tilde{C}} - 1}\right). \quad (35)\end{aligned}$$

the rest of the proof follows the footsteps of the proof of Lemma 1. ■

Theorem 3: For a symmetric N-user $N_r \times N_t$ Rayleigh-fading MIMO-MAC, we have

$$Pr\left(C_{\Sigma-\text{sym}} < R|\tilde{C}\right) \leq \sum_{k=1}^{N} \binom{N}{k} \tilde{P}_{\text{out}}(k, R|\tilde{C}) \quad (36)$$

Proof: By (29) and (30), for every $\mathcal{S}$ and channel realization, it holds that

$$\tilde{C}(\mathcal{S}) \leq C(\mathcal{S}). \quad (37)$$

Denoting

$$\begin{aligned}\tilde{C}_{\text{sym}} &= \min_{\mathcal{S}\subseteq\{1,2,\ldots,N\}} \frac{N}{|\mathcal{S}|} \log\left(1 + \|\mathcal{H}_S\|_F^2\right) \\ &= \min_{\mathcal{S}\subseteq\{1,2,\ldots,N\}} \frac{N}{|\mathcal{S}|} \tilde{C}(\mathcal{S}), \quad (38)\end{aligned}$$

we have

$$\Pr\left(C_{\Sigma-\text{sym}} < R \middle| \tilde{C}\right) \leq \Pr\left(\tilde{C}_{\text{sym}} < R \middle| \tilde{C}\right), \quad (39)$$

and similar to Theorem 1, applying the union bound, we get

$$\Pr\left(\tilde{C}_{\text{sym}} < R \middle| \tilde{C}\right) \leq \sum_{k=1}^{N} \binom{N}{k} \tilde{P}_{\text{out}}(k, R|\tilde{C}). \quad (40)$$

■

Figure 7 depicts a comparison between the empirical outage probability and the upper bound provided by Theorem 3 for the case of two users, each equipped with 2 antennas and a receiver equipped with 3 antennas, where the target rate is set to 3 bits. The outage probability is depicted as a function of the SNR. The outage probability was evaluated empirically by Monte-Carlo simulation. To calculate the bound, the Frobenius norm of each channel matrix drawn was calculated. We recall that when defining the channel model we assumed that the SNR is absorbed in the channel gains (i.e., that the channel is distributed as $\mathbf{H}_i \sim \mathcal{CN}(0, \mathsf{SNR}\cdot\mathbf{I}_{N_r})$) where now we assume that the channel is distributed according to the standard i.i.d. Rayleigh-fading model ($\mathbf{H}_i \sim \mathcal{CN}(0, \cdot\mathbf{I}_{N_r})$).

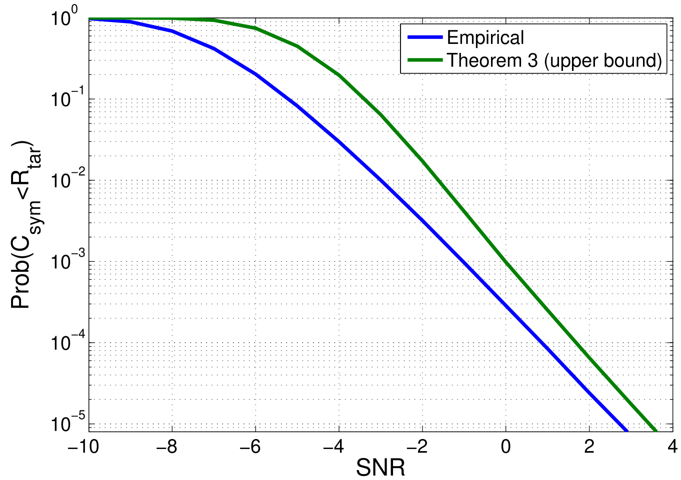

Fig. 7. Comparison of empirical outage and the upper bound provided by Theorem 3 for a two-user 3×2 i.i.d. Rayleigh-fading MAC. The target rate is set to 3 bits.

It can seen that at high SNR, the slope of the bound is similar to that of the empirical results. Recalling the MIMO-MAC DMT, we note that since the target rate is fixed (is not a function of the SNR), the slope at high SNR is in fact the maximal diversity offered in this configuration, i.e., the diversity corresponding to zero multiplexing gain. Recalling the DMT of the symmetric capacity (7), the latter is $N \cdot N_r \cdot N_t$ which matches the slope given by Theorem 2. On the other hand, relying on the Frobenius norm results in a loose bound at low values of SNR (high outage probabilities).

We may obtain a tighter bound for low SNR values, for the special case of $N = 2$ users, where each is equipped with a single antenna and the receiver is equipped with $N_r \geq 2$ antennas. Specifically, in [13] a different upper bound for the outage probability was derived in the context of a randomly precoded compound single-user $N_r \times 2$ MIMO channel. It is easy to verify that the derived bound carries over to the setting considered in the present paper, when rewritten as follows:[4]

Theorem 4 (Theorem 2 in [13]): For a two-user i.i.d. Rayleigh-fading MAC where each user is equipped with a single antenna and the receiver is equipped with N_r antennas,

$$Pr\left(C_{\Sigma-\text{sym}} < R|C\right) \leq 1 - \sqrt{1 - 2^{-(C-R)}}. \quad (41)$$

The main advantage Theorem 4 with respect to Theorem 3 is that the conditioning is on C, the true sum capacity of the channel, rather than its Frobenius-norm counterpart.

[4]The main step is to recall that the SVD decomposition of an i.i.d. circularly-symmetric complex Gaussian matrix yields left and right singular vector matrices that are uniformly (Haar) distributed, as is the precoding matrix considered in the analysis of [13].

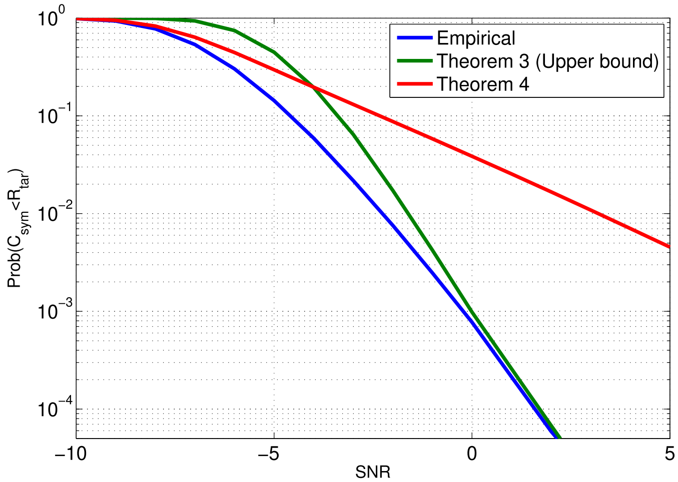

Fig. 8. Comparison of empirical outage and the upper bound provided by Theorems 3 and 4 for a symmetric two-user 6×1 i.i.d. Rayleigh-fading MAC. The target rate is set to 3 bits.

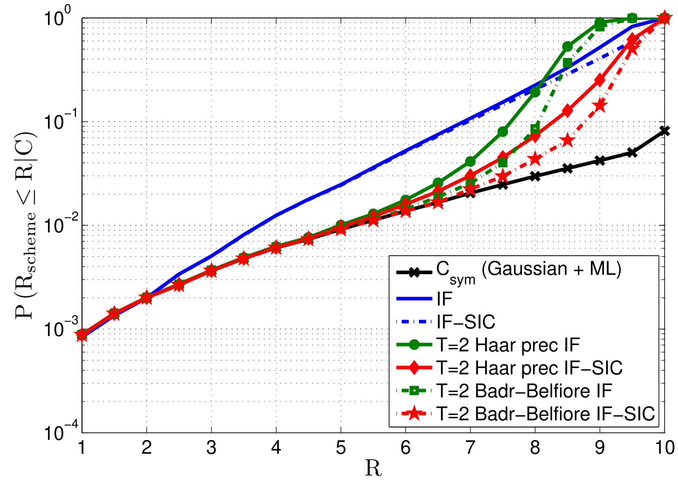

Fig. 9. Outage probability of linear codes (with and without space-time precoding) with IF equalization versus Gaussian codebooks with ML decoding for a two-user i.i.d. Rayleigh-fading MAC with sum capacity $C = 10$.

Figure 8 depicts the empirical outage probability, the upper bound of Theorem 3 and the bound of Theorem 4 for the case of a symmetric two-user 6×1 Rayleigh-fading MAC, where the target rate is set to 3 bits. It can be seen that at low SNR, Theorem 4 provides a tighter bound than Theorem 3 but it becomes loose rapidly as it does not capture the maximal diversity offered by the system.

VI. Practical Realization of the Communication Protocol via Precoded Integer Forcing

In this section we empirically demonstrate the effectiveness of the integer-forcing (IF) receiver when used in conjunction with unitary space-time precoding as a practical transmission scheme in the context of the considered communication protocol, performance being evaluated assuming long blocklength. We refer the reader to [14] for a description of the integer forcing framework and its implementation.

We note that IF has significantly lower complexity compared to maximum-likelihood detection as its complexity does not scale (exponentially) with the block size. We further note that the integer-forcing architecture is applicable for all blocklength, starting from blocklength one (uncoded transmission, in which case it reduces to lattice reduction-aided decoding). In particular, there are several works which study the performance of IF with practical codes of finite block size including [15], [16].

In this section, we highlight several potential improvements for standard IF and show that when combined with standard IF, it results with significant improvement. As the suggested improvements does not depend on the block size, analyzing performance of IF with short block size for the MAC channel is left for further study.

When it comes to fading channels, it has been shown in [14] that the IF receiver achieves the DMT over i.i.d. Rayleigh-fading channels where the number of receive antennas is greater or equal to the number of transmit antennas.

We observe that this does not hold in the general case; in particular, IF does not achieve the DMT for the case of a MAC where all terminals are equipped with a single antenna. Specifically, Figure 9 depicts (in logarithmic scale) the empirical outage probability of the IF receiver and the exact outage probability for optimal communication (Gaussian codebooks and ML decoding), as given by Theorem 1, for the two-user i.i.d. Rayleigh-fading MAC. The symmetric rate achieved by a given scheme is denoted by R_{scheme}.

It is evident that the slopes are different. This raises the question of whether IF is inherently ill-suited for the MAC channel. A negative answer to this question may be inferred by recalling some further lessons from the DMT analysis of the MAC.

While the optimal DMT for the i.i.d. Rayleigh-fading MAC was derived in [2] using Gaussian codebooks of sufficient length, it was subsequently shown that the DMT of the MAC can be achieved using structured codebooks by combining uncoded QAM constellations with space-time unitary precoding (and ML decoding). Specifically, such a MAC-DMT achieving construction is given in [5]. This raises the possibility that the sub-optimality of the IF receiver observed in Figure 9 may at least in part be remedied by applying unitary space-time precoding at each of the transmitters. We note that each transmitter applies precoding only to its own data streams so the distributed nature of the problem is not violated.

Following this approach, we have implemented the IF receiver with unitary space-time precoding applied at each transmitter. We have employed random (Haar) precoding (with independent matrices drawn for the different users) over two ($T = 2$) time instances as well as deterministic precoding using the matrices proposed in [17].[5]

These matrices can be expressed as

$$\mathbf{P}^1_{st,c} = \frac{1}{\sqrt{5}} \begin{bmatrix} \alpha & \alpha\phi \\ \bar{\alpha} & \bar{\alpha}\bar{\phi} \end{bmatrix}, \quad \mathbf{P}^2_{st,c} = \frac{1}{\sqrt{5}} \begin{bmatrix} j\alpha & j\alpha\phi \\ \bar{\alpha} & \bar{\alpha}\bar{\phi} \end{bmatrix} \qquad (42)$$

[5]When using an ML receiver, this space-time code is known to achieve the DMT for multiplexing rates $r \leq \frac{1}{5}$. As detailed in [5], whether this code achieves the optimal MAC-DMT also when $r > \frac{1}{5}$ remains an open question.

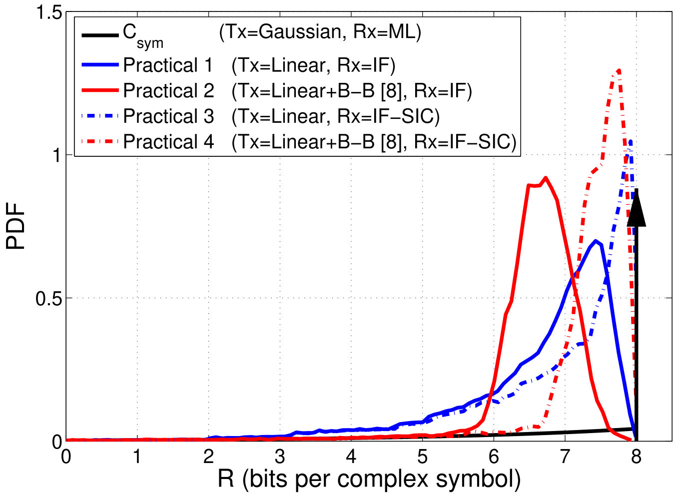

Fig. 10. Probability distribution function of the rate achieved with linear codes (with and without space-time precoding) in conjunction with IF equalization versus that achieved Gaussian codebooks with ML decoding for a two-user i.i.d. Rayleigh-fading MAC with sum capacity $C = 10$.

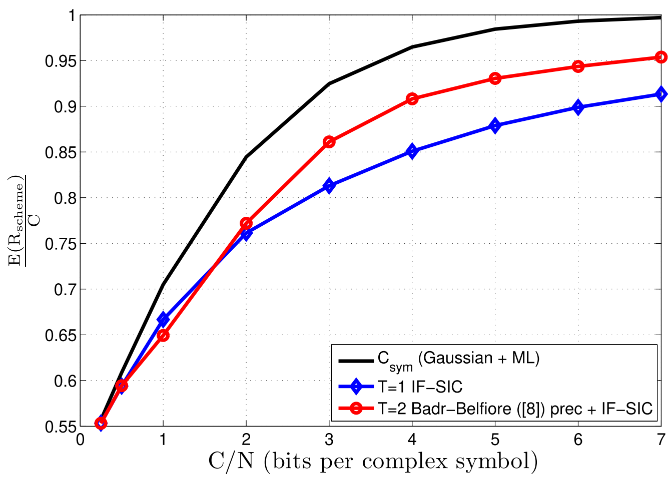

Fig. 11. Average rate conditioned on the sum capacity when using linear codes (with and without space-time precoding) with IF equalization versus Gaussian codebooks with ML decoding, over a two-user normalized (conditioned) i.i.d. Rayleigh-fading MAC.

where

$$\phi = \frac{1+\sqrt{5}}{2}, \quad \bar{\phi} = \frac{1-\sqrt{5}}{2}$$
$$\alpha = 1 + j - j\phi, \quad \bar{\alpha} = 1 + j - j\bar{\phi}. \tag{43}$$

We also replot Figure 9 in terms of PDF (rather than CDF) as Figure 10, but without random Haar space-time precoding (so as to avoid "clutter"). As can be seen, the precoding matrices in (42) improve the outage probability for most target rates.

We further note that in addition to standard IF, we also implemented a variant that incorporates successive interference cancellation, referred to as IF-SIC [18]. As can be seen, IF-SIC results in a significant improvement for all precoding schemes used.

In Figure 11 we study the average symmetric rate achieved by different schemes w.r.t. a two-user i.i.d. Rayleigh-fading channel when we condition on the sum capacity of the channel.

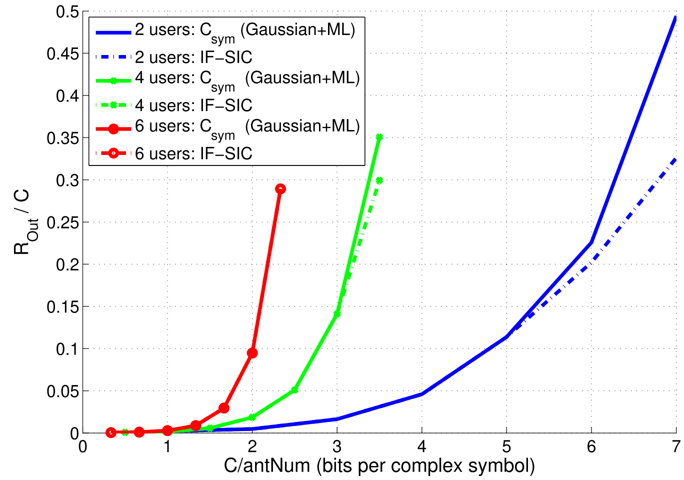

Fig. 12. Fraction of the sum capacity achieved at 1% outage probability by the proposed transmission protocol over a scalar i.i.d. Rayleigh-fading MAC. The performance limits, as captured by by C_{sym} is depicted as a function of the sum capacity normalized by the number of users, for $N = 2, 4, 6$ users. The performance limits of IF-SIC equalization are also depicted.

We plot the fraction of the sum capacity attained by the various schemes. We first observe that IF-SIC combined with space-time precoded linear codes achieves a large fraction of the symmetric capacity. Further, as can be seen, the fraction of the sum capacity achieved by all the different schemes considered approaches one as the sum capacity grows.

Finally, in Figure 12, we plot the fraction of the sum capacity that is achieved, allowing for a fixed outage probability, by the proposed protocol where we consider both the ideal performance achieved as captured by the symmetric capacity and the rate achieved using IF in conjunction with SIC. As can be observed, the performance of IF-SIC for small outage probabilities is very close to the theoretical limits of the considered transmission protocol. We note, however, that as the number of users increases (and also, as the sum-capacity increases), the problem of finding a "good" integer matrix (as required in IF equalization) becomes computationally difficult and may result in compromised rates when using practical sub-optimal algorithms such as the LLL algorithm to find candidate integer matrices.

VII. Conclusion

We analyzed the performance of a simple communication protocol for transmission over the scalar Rayleigh-fading MAC, where all users transmit at just below the symmetric capacity (normalized per user) of the channel. Tight bounds were established on the distribution of the achievable rate of the protocol. The derived bounds may be viewed as a significant tightening of the diversity multiplexing tradeoff analysis of the channel. It was further demonstrated that integer-forcing equalization in conjunction with "space-time" precoding (over the time axis only) offers a practical means to approach the theoretical limits of the proposed protocol.

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